

# q-ary Compressive Sensing

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## Abstract

We introduce  $q$ -ary compressive sensing, an extension of 1-bit compressive sensing. We propose a novel sensing mechanism and a corresponding recovery procedure. The recovery properties of the proposed approach are analyzed both theoretically and empirically. Results in 1-bit compressive sensing are recovered as a special case. Our theoretical results suggest a tradeoff between the quantization parameter  $q$ , and the number of measurements  $m$  in the control of the error of the resulting recovery algorithm, as well its robustness to noise.

## 1 Introduction

Reconstructing signals from discrete measurements is a classic problem in signal processing. Properties of the signal inform the way reconstruction can be achieved from a minimal set of measurements. The classical Shannon sampling result ensures that band limited signals can be reconstructed by a linear procedure, as long as a number of linear measurements, at least twice the maximum frequency, is available. Modern data analysis typically requires recovering high dimensional signals from few inaccurate measurements. Indeed, the development of Compressed Sensing (CS) and Sparse Approximation [2] shows that this is possible for signals with further structure. For example,  $d$ -dimensional,  $s$ -sparse signals<sup>1</sup> can be reconstructed with high probability through convex programming, given  $m \sim s \log(d/s)$  random linear measurements.

Non linear measurements have been recently considered in the context of 1-bit compressive sensing (<http://dsp.rice.edu/1bitCS/>). Here, binary (one-bit) measurements are obtained by applying, for example, the “sign” function<sup>2</sup> to linear measurements. More precisely, given  $x \in \mathbb{R}^d$ , a measurement vector is given by  $y = (y_1, \dots, y_m)$ , where  $y_i = \text{sign}(\langle w_i, x \rangle)$  with  $w_i \sim \mathcal{N}(0, I_d)$  independent Gaussian random vectors, for  $i = 1, \dots, m$ . It is possible to prove [1] that, for a signal  $x \in K \cap \mathbb{B}^d$  ( $\mathbb{B}^d$  is the unit ball in  $\mathbb{R}^d$ ), the solution  $\hat{x}_m$  to the problem

$$\max_{x \in K} \sum_{i=1}^m y_i \langle w_i, x \rangle, \quad (1)$$

satisfies  $\|\hat{x}_m - x\|^2 \leq \frac{\delta}{\sqrt{\frac{2}{\pi}}}$ , with probability  $1 - 8 \exp(-c\delta^2 m)$ ,  $\delta > 0$ , as long as  $m \geq C\delta^{-2}\omega(K)^2$  [1]. Here,  $C$  denotes a universal constant and  $\omega(K) = \mathbb{E} \sup_{x \in K-K} \langle w, x \rangle$  the Gaussian mean width  $K$ , which can be interpreted as a complexity measure. If  $K$  is a convex set, problem (1) can be solved efficiently.

<sup>1</sup>A  $d$ -dimensional signal, that is a vector in  $\mathbb{R}^d$ , is  $s$ -sparse if only  $s$  of its components are different from zero.

<sup>2</sup>More generally, any function  $\theta : \mathbb{R} \rightarrow [-1, 1]$ , such that  $\mathbb{E}(g\theta(g)) > 0$  can be used.

In this paper, borrowing ideas from signal classification studied in machine learning, we discuss a novel sensing strategy, based on  $q$ -ary non linear measurements, and a corresponding recovery procedure.

## 2 $q$ -ary Compressive Sensing

In this section we first describe the sensing and recovery procedure (Section 2.1), then describe the results in the noiseless (Section 2.2) and noisy setting (Section 2.3), and finally, we sketch the main ideas of the proof (Section 2.4).

### 2.1 Sensing and Recovery

The sensing procedure we consider is given by a map  $C$  from  $K \cap \mathbb{B}^d$  to the  $q$ -ary,  $m$  Hamming cube  $\{0, \dots, q-1\}^m$ , where  $K \subset \mathbb{R}^d$ . To define  $C$  we need the following definitions.

**Definition 1** (Simplex Coding [3]). *The simplex coding map is  $S : \{0, \dots, q-1\} \rightarrow \mathbb{R}^{q-1}$ ,  $S(j) = s_j$ , where*  
1)  $\|s_j\|^2 = 1$ ,  
2)  $\langle s_j, s_i \rangle = -\frac{1}{q-1}$ , for  $i \neq j$ ,  
3)  $\sum_{j=0}^{q-1} s_j = 0$ ,  
*for all  $i, j = 0, \dots, q-1$ .*

**Definition 2** ( $q$ -ary Quantized Measurements). *Let  $W \in \mathbb{R}^{q-1, d}$  be a Gaussian random matrix, i.e.  $W_{ij} \sim \mathcal{N}(0, 1)$  for all  $i, j$ . Then,  $Q : K \cap \mathbb{B}^d \rightarrow \{0, \dots, q-1\}$ ,*

$$Q(x) = Q_W(x) = \arg \max_{j=0 \dots q-1} \langle s_j, Wx \rangle,$$

*is called a  $q$ -ary quantized measurement.*

Then, we can define the  $q$ -ary sensing strategy induced by non linear quantized measurements.

**Definition 3** ( $q$ -ary Sensing). *Let  $W_1, \dots, W_m$ , be independent Gaussian random matrices in  $\mathbb{R}^{q-1, d}$  and  $Q_{W_i}(x)$ ,  $i = 1, \dots, m$  as in Def. 2. The  $q$ -ary sensing is  $C : K \cap \mathbb{B}^d \rightarrow \{0, \dots, q-1\}^m$ ,*

$$C(x) = (Q_{W_1}(x), \dots, Q_{W_m}(x)),$$

$\forall x \in K \cap \mathbb{B}^d$ .

Before describing the recovery strategy we consider, we add two remarks.

**Remark 1** (Connection to 1-bit CS). *If  $q = 2$ ,  $W$  reduces to a Gaussian random vector, and  $2Q(x) - 1 = \text{sign}(Wx)$ , so that the  $q$ -ary quantized measurements become equivalent to those considered in 1-bit CS.*

**Remark 2** (Sensing and Embeddings). *It can be shown that  $C$  defines an  $\epsilon$ -isometric embedding of  $(K, \|\cdot\|)$ , into  $(\mathcal{F}, d_H)$  – up-to a bias term. Here  $d_H$  is the (normalized) Hamming distance,  $d_H(u, v) = \frac{1}{m} \sum_{i=1}^m \mathbb{1}_{u_i \neq v_i}$ ,  $u, v \in \mathcal{F}$ . This analysis is deferred to the long version of this paper.*

In this paper, we are interested in provably (and efficiently) recovering a signal  $x$  from its  $q$ -ary measurements  $y = (y_1, \dots, y_m) = C(x)$ . Following [1], we consider the recovery strategy  $D : \{0, \dots, q-1\}^m \rightarrow K \cap \mathbb{B}^d$  defined by,

$$D(y) = \arg \max_{u \in K \cap \mathbb{B}^d} \frac{1}{m} \sum_{i=1}^m \langle s_{y_i}, W_i u \rangle. \quad (2)$$

The above problem is convex as soon as  $K$  is convex and can be solved efficiently, see Section 3.1. In the next section, we prove it has good recovery guarantees both in noiseless and noisy settings. We first add a remark.

**Remark 3** (Connection to Classification). *An inspiration for considering the  $q$ -ary CS stems from an analogy between 1-bit compressed sensing and binary classification in machine learning. In this view, Definition (3) is related to the approach proposed for multi category classification in [3]. Following these ideas, we can extend the recovery strategy (2) by considering*

$$D_V(y) = \arg \min_{u \in K \cap \mathbb{B}^d} \frac{1}{m} \sum_{i=1}^m V(-\langle s_{y_i}, W_i u \rangle), \quad (3)$$

where  $V$  is a convex, Lipchitz, non decreasing loss function  $V : \mathbb{R} \rightarrow \mathbb{R}^+$ . Problem (2) corresponds to the choice  $V(x) = x$ . Other possible choices include  $V(x) = \max(1 + x, 0)$ ,  $V(x) = \log(1 + e^x)$ , and  $V(x) = e^x$ . All these loss functions can be seen as convex relaxations of the 0-1 loss function, defined as  $V(x) = 0$  if  $x \leq 0$ , and 1 otherwise. The latter defines the misclassification risk, which corresponds to Hamming distance in CS, which is the natural measure of performance while learning classification rules.

## 2.2 Recovery guarantees: Noiseless Case

The following theorem describes the recovery guarantees for the proposed procedure, when applied on a signal  $x$  in a set  $K$  of Gaussian mean width  $w(K)$ . We first consider a noiseless scenario.

**Theorem 1.** *Let  $\delta > 0$ , and  $m \geq C\delta^{-2}w(K)^2$ . Then with probability at least  $1 - 8\exp(-c\delta^2m)$ , the solution  $\hat{x}_m = D(y)$  of problem (2) satisfies,*

$$\|\hat{x}_m - x\|^2 \leq \frac{\delta}{\sqrt{\log(q)}}. \quad (4)$$

A proof sketch of the above result is given in Section 2.4, while the complete proof is deferred to the long version of the paper. Here, we add four comments. First, we note that the above result implies the error bound,

$$\|\hat{x}_m - x\|^2 \leq C\left(\frac{w(K)}{\sqrt{\log(q)m}} + \delta\right), \quad (5)$$

with probability at least,  $1 - 4\exp(-2\delta^2)$ ,  $\delta > 0$ .

Second, Inequalities (4), (5) can be compared to results in 1-bit CS. For the same number of measurements,  $m \geq C\delta^{-2}w(K)^2$ , the error for  $q$ -ary CS is  $\frac{\delta}{\sqrt{\log(q)}}$ , in contrast with  $\frac{\delta}{\sqrt{\frac{2}{\pi}}}$  in the 1-bit CS [1], at the expense of a more demanding sensing procedure. Also note that, for  $q = 2$ , we recover the result in 1-bit CS as a special case. Third, we see that for a given accuracy our results highlights a trade-off between the number of  $q$ -ary measurements  $m$  and the quantization parameter  $q$ . To achieve an error  $\epsilon$ , with a memory budget of  $\ell$  bits, one can choose  $m$  and  $q$  so that  $\epsilon = O(\frac{1}{\sqrt{m \log(q)}})$ , and  $m \log_2(q) = \ell$  (see also section 3.2). Finally, in the following we will be interested in  $K$  being the set of  $s$ -sparse signals. Following again [1], it is interesting to consider in Problem (2) the relaxation

$$K_1 = \{u \in \mathbb{R}^d : \|u\|_1 \leq \sqrt{s}, \|u\|_2 \leq 1\}.$$

With this choices, it is possible to prove that  $w(K_1) \leq C\sqrt{s \log(\frac{2d}{s})}$ , and that for  $m \geq C\delta^{-2}s \log(\frac{2d}{s})$ , the solution of the convex program (2) on  $K_1$  satisfies,  $\|\hat{x}_m - x\|^2 \leq \frac{\delta}{\sqrt{\log(q)}}$ . We end noting

that other choices of  $K$  are possible, for example in [6] the set of group sparse signals (and their Gaussian width) are studied.

### 2.3 Recovery Guarantees: Noisy Case

Next we discuss the  $q$ -ary approach in two noisy settings, related to those considered in [1].

**Noise before quantization.** For  $i = 1, \dots, m$ , let

$$y_i = \arg \max_{j=0 \dots q-1} \{ \langle s_j, W_i x \rangle + g_j \}, \quad (6)$$

with  $g_j$  independent Gaussian realization of variance  $\sigma^2$ . In this case, it is possible to prove that, for  $m \geq C\delta^{-2}w(K)^2$ ,

$$\|\hat{x}_m - x\|^2 \leq \frac{\delta\sqrt{1+\sigma^2}}{\sqrt{\log(q)}},$$

with probability at least  $1 - 8 \exp(-c\delta^2 m)$ . The quantization level  $q$  can be chosen to adjust to the noise level  $\sigma$  for a more robust recovery of  $x$ . This result can be viewed in the perspective of the *bit-depth versus measurement-rates* perspective studied in [4]. Here it is shown that 1-bit CS outperforms conventional scalar quantization. In this view,  $q$ -ary CS provides a new way to adjust the quantization parameter to the noise level.

**Inexact maximum.** For  $i = 1, \dots, m$ , let  $y_i = Q_{W_i}(x)$ , with probability  $p$ , and  $y_i = r$  with probability  $1 - p$ , with  $r$  drawn uniformly at random from  $\{0, \dots, q - 1\}$ . In this case, it is possible to prove that, for  $m \geq C\delta^{-2}w(K)^2$ ,

$$\|\hat{x}_m - x\|^2 \leq \frac{\delta}{\sqrt{\log(q)}(2p - 1)}.$$

with probability at least  $1 - 8 \exp(-c\delta^2 m)$ . The signal  $x$  can be recovered even if *half* of the  $q$ -ary bits are flipped.

### 2.4 Elements of the proofs

We sketch the main steps in proving our results. The proof of Theorem 1 is based on: 1) deriving a bound in expectation, and 2) deriving a concentration result. The proof of the last step uses Gaussian concentration inequality extending the proof strategy in [1]. Step 1) gives the bound

$$\mathbb{E}(\|\hat{x}_m - x\|^2) \leq \frac{w(K)}{C\sqrt{\log(q)m}},$$

the proof of which is based on the following proposition.

**Proposition 1.** Let  $\mathcal{E}_x(u) = \mathbb{E}_W(\langle s_\gamma, Wu \rangle)$ , where  $\gamma = Q_W(x)$ . Then,  $\forall u \in \mathbb{B}^d$ , we have,

$$\frac{1}{2} \|u - x\|^2 \leq \frac{1}{\lambda(q)} (\mathcal{E}_x(x) - \mathcal{E}_x(u)),$$

where  $\lambda(q) = \mathbb{E}_{\bar{\gamma}, g}(\langle s_{\bar{\gamma}}, g \rangle)$ , and  $g \sim \mathcal{N}(0, I_{q-1})$ , and  $\bar{\gamma} = \arg \max_{j=0 \dots q-1} \langle s_j, g \rangle$ .

Using results in empirical process theory it possible to show that

$$|\mathcal{E}_x(x) - \mathcal{E}_x(\hat{x}_m)| \leq C \frac{w(K)}{\sqrt{m}}.$$

The bound on the expected recovery follows combining the above inequality and Proposition 1 with the inequality,

$$\lambda(q) \geq C\sqrt{\log(q)},$$

which is proved using Slepian inequality and Sudakov minoration.

The results in the noisy settings follow from suitable estimates of  $\lambda(q)$ . Indeed, for the *noise before quantization* case it can be proved that  $\lambda(q) \geq C\sqrt{\frac{\log(q)}{1+\sigma^2}}$ . For the *inexact maximum* case one has

$$\begin{aligned}\lambda(q) &= \mathbb{E}_{y,g}(\langle s_y, g \rangle) = \\ &= p\mathbb{E}(\max_{j=1\dots q} \langle s_j, g \rangle) + (1-p)\mathbb{E}(\langle s_r, g \rangle) \geq \\ &= Cp\sqrt{\log(q)} + (1-p)\mathbb{E}(\min_{j=1\dots q} \langle s_j, g \rangle) \geq \\ &= (2p-1)C\sqrt{\log(q)}.\end{aligned}$$

### 3 Experimental Validation

In this section, we describe some numerical simulations in sparse recovery, Section 3.1, and preliminary experiments in an image recovery problem, Section 3.3.

#### 3.1 An Algorithm for Sparse recovery

In our experiments, we considered the following variation of problem (2), Let  $\xi_i = s_{y_i}^\top W_i \in \mathbb{R}^d, i = 1 \dots m$ .

$$\max_{u, \|u\|_2 \leq 1} \frac{1}{m} \sum_{i=1}^m \langle \xi_i, u \rangle - \eta \|u\|_1, \quad (7)$$

where  $\eta > 0$ . The above problem can be solved efficiently using Proximal Method [5], a solution can be computed via the iteration,

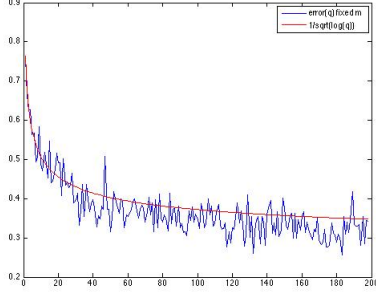
$$\begin{aligned}u_{t+1} &= u_t + \frac{\nu_t}{m} \sum_{i=1}^m \xi_i, \\ u_{t+1} &= \text{Prox}_\eta(u_{t+1}), \\ u_{t+1} &= u_{t+1} \min\left(\frac{1}{\|u_{t+1}\|_2}, 1\right).\end{aligned}$$

Where  $\nu_t$  is the gradient step size, and  $\text{Prox}_\eta$  acts component wise as  $\max(1 - \frac{\eta}{|u_i|}, 0)u_i$ . The iteration is initialized randomly to a unit vector.

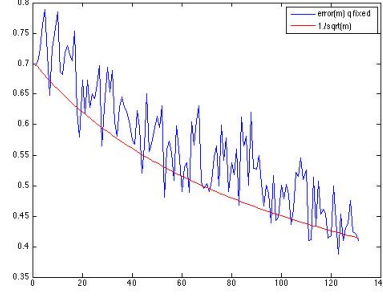
**Remark 4.** The computational complexity of the sensing process depends on both  $m$  and  $q$ . Whereas, the computational complexity of the recovery algorithm, once computed  $\xi_i$ , is independent to the choice of  $q$ , and depends only on  $m$  and is the same as in 1-bit CS.

#### 3.2 Sparse Recovery

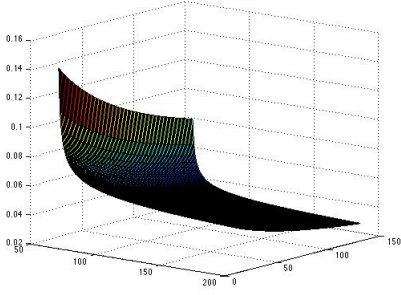
We tested our approach for recovering a signal from its  $q$ -ary measurements. We considered sparse signals of dimension  $d$  generated via a Gauss-Bernoulli model. In Figure 1(a), we see that the reconstruction error of  $\hat{x}_m$  (in blue), for varying  $q$  and  $m$  fixed, follows the theoretical bound  $\frac{1}{\sqrt{\log(q)}}$  (in red). In Figure 1(b), we see that the reconstruction error  $\hat{x}_m$  (in blue), for varying  $m$  and  $q$  fixed, follows the theoretical bound  $\frac{1}{\sqrt{m}}$  (in red). Figures 1(c), and 1(d) highlight the tradeoff between the number of measurements and the quantization parameter. For a precision  $\epsilon$ , and a memory budget  $2^B$ , one can choose an operating point  $(m, q)$ , according to the theoretical bound  $\frac{1}{\sqrt{m \log(q)}}$ .



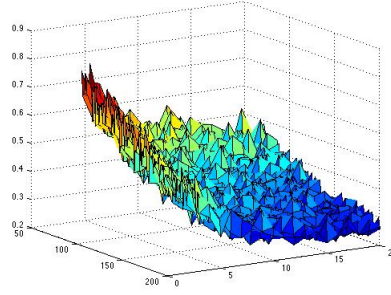
(a) Error  $\|x - \hat{x}\|^2$  versus  $q$ , for  $m = 70, d = 100$ .



(b) Error  $\|x - \hat{x}\|^2$  versus  $m$ , for  $q = 3, d = 100$ .



(c) Theoretical bound for  $\|x - \hat{x}\|^2$  versus  $m$  and  $q$ .



(d) Error  $\|x - \hat{x}\|^2$  versus  $m$  and  $q$ .

Figure 1:  $q$ -ary Compressive Sensing: Quantization/Number of measurements tradeoff.

### 3.3 Image Reconstruction

Then, we considered the problem of recovering an image from  $q$ -ary measurements. We used the standard 8-bit grayscale boat image of size  $64 \times 64$  pixels shown in Figure 2(a). We extracted the wavelet coefficients and performed thresholding to get a sparse signal. We normalized the resulting vector of wavelets coefficients of dimension  $d = 3840$  to obtain a unit vector. Then, we performed sensing and recovery with  $q = 2^5$  (5-bit compressive sensing) and  $q = 2$  (1-bit compressive sensing) for the same  $m = 2048 < d$ . We compared the SNR performances of the corresponding reconstructed images in a noiseless setting (Figures 2(b)-(c)), and a noisy setting, considering the noise before quantization model (6), with  $\sigma = 0.8$  (Figures 2(d)-(e)). The results confirm our theoretical: higher quantization improves the SNR, as well as the robustness to noise of  $q$ -ary compressive sensing.

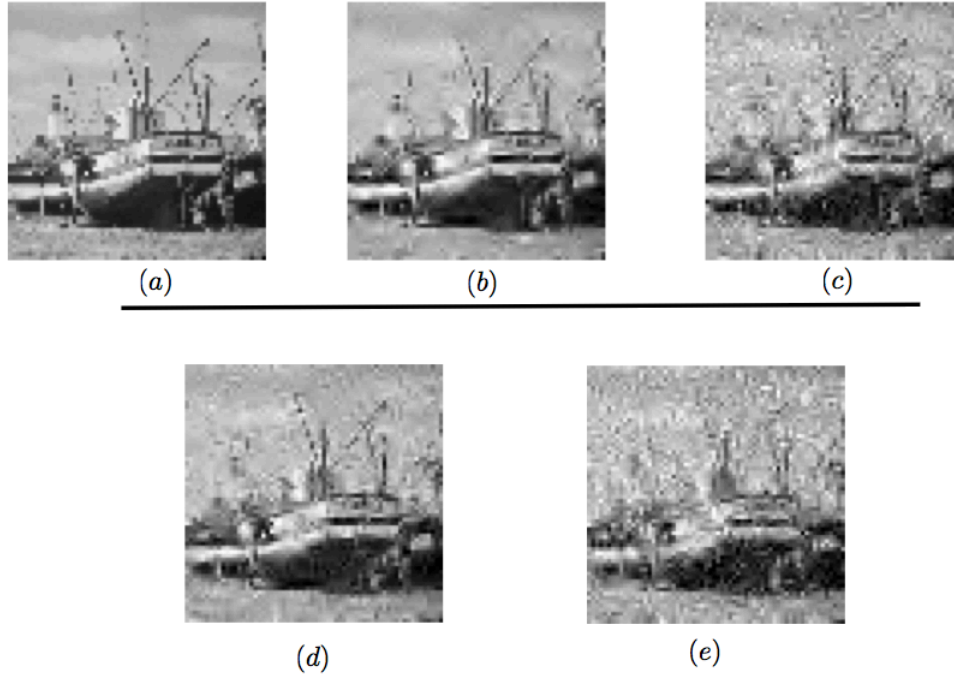


Figure 2: Image recovery with  $q$ -ary CS. (a) Original image. (b) Reconstruction with no-noise:  $q = 2^5$ , SNR = 20.2 dB. (c) Reconstruction with no-noise:  $q = 2$ , SNR = 16.2 dB. (d) Reconstruction with noise:  $q = 2^5$ , SNR = 18.3 dB. (e) Reconstruction with noise:  $q = 2$ , SNR = 15 dB.

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